

## Exact solution of a one-dimensional deterministic sandpile model

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Using the transfer matrix method, we give the exact solution of a deterministic sandpile model for arbitrary  $N$ , where  $N$  is the size of a single toppling. The one- and two-point functions are given in term of the eigenvalues of an  $N \times N$  transfer matrix. All the  $n$ -point functions can be found in the same way. Application of this method to a more general class of models is discussed. We also present a quantitative description of the limit cycle (attractor) as a multifractal.

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### I. INTRODUCTION

The sandpile model has been conceived as the simplest model which can illustrate self-organized criticality (SOC). It has developed into an interesting cellular automata model of nonlinear dynamics in its own right. Bak, Tang, and Wiesenfeld [1,2] were the first to use numerical methods to study some of the sandpile models and observed that the models automatically evolve into a self-organized critical state while they possess  $1/f$  spectra in both spatial and temporal distributions of certain physical quantities. It was conjectured that SOC may be a universal characteristic underlying nonlinear, dispersive systems in nature such as earthquakes, forest fire, turbulence, etc.

The simplest cellular automata model of the sandpile is to assign a height number  $h_i$  to each site on a one-dimensional lattice with length  $L$ . There are two basic operations of the model—dropping and toppling. Dropping means that one sand grain is added at some site of the lattice, i.e.,  $D_i: h_i \rightarrow h_i + 1$ . Toppling occurs when a slope (defined as the difference in height between adjacent sites) exceeds some critical value  $N$ . If toppling occurs at one site, some sand grains at the site will be moved to other sites which may trigger further topplings.

Even with these simple rules of evolution, the analytic solution of the sandpile model is typically very difficult to come by especially when  $L$  or  $N$  becomes large, or when the lattice dimension is more than 1. Most researchers handle the models by numerical simulation [3–5]. If the rules are such that the evolution of the system is independent of the order of the droppings, then the model is called Abelian. For a large class of Abelian models some exact results have been obtained by Dhar and Ramaswamy [6]. The non-Abelian models are typically much harder to solve. Here we shall investigate a class of non-Abelian models in the deterministic limit, i.e., the sand being dropped at a fixed site [7].

We consider the one-dimensional case and label the sites from left to right as 1 to  $L$ . The sand is dropped only at the site 1. If the slope at a site exceeds a given

number  $N$ , then the sand will topple to the right. Let the slope  $\sigma_i = h_i - h_{i+1}$ ; then the toppling rule is “if  $\sigma_i > N$ , then  $\sigma_{i-1} \rightarrow \sigma_{i-1} + N$ ,  $\sigma_i \rightarrow \sigma_i - (N + 1)$ , and  $\sigma_{i+N} \rightarrow \sigma_{i+N} + 1$ .” The rule should be modified when toppling occurs near the boundary. The condition at the left boundary is trivial. When sand grains reach beyond the right boundary they drop out from the system, i.e., we keep  $h_i = 0$  for  $i > L$ . A state in which all  $\sigma_i \leq N$  is called a stable state. Toppling stops when a stable state is reached. Each dropping and subsequent toppling process will result in the transition from one stable state to another. Since there are only a finite number of different states in the system, after dropping enough sand at site 1, the system will step into a cycle called the limit cycle. For the system that we consider here there is only one limit cycle in the problem. Not all the stable states are in the limit cycle. Those in the limit cycle are called allowed states. In the following, all the states we will refer to are allowed states. The number of allowed states in the limit cycle is  $N^L$  [8].

Some special solvable non-Abelian models have been investigated in Refs. [8–10]. In Ref. [8], the simplest ( $N = 2$ ) of a class of non-Abelian sandpile models was solved. Some general properties of the model for arbitrary  $N$  were also exposed including some results about the random drop cases [9]. In Ref. [10] a new method using the transfer matrix idea from statistical mechanics models was developed and applied to solve the  $N = 3$  case of the same class of sandpile models. It was commented that the method should also be useful for solving models with arbitrary  $N$ . In this note, we wish to develop this idea further and supply a solution of the model with arbitrary  $N$ .

The method to be introduced here is particularly useful for those models in which the structure of the limit cycles has been worked out. Once the cycle structure is known, its information can be succinctly summarized in a matrix which we call the transfer matrix by analogy with the similar matrix in statistical mechanics. For the deterministic model defined above, the structure of the limit cycle has been worked out in Ref. [8]. Therefore we

should use it as our main example; the method may be applicable to much wider classes of models. A multifractal description of the limit cycle is also worked out.

## II. THE DEFINITION OF THE TRANSFER MATRIX

The allowed states of length  $L$  in the limit cycle are characterized by three conditions.

- (1)  $\sigma_i \geq 0$  for  $0 < i \leq L$ .
- (2) There exists at least one site  $i$  for any consecutive  $N$  sites such that  $\sigma_i = N$  (stability condition).
- (3) There exists a site  $i$  satisfying  $L - \sigma_L \leq i < L$  such that  $\sigma_i = N$  (boundary condition).

If  $[\sigma_1 \sigma_2 \dots \sigma_L]$  is an allowed state in the limit cycle with length  $L$ , then  $[\sigma_{i+1} \sigma_{i+2} \dots \sigma_L]$  is also an allowed state in the limit cycle with length  $L - i$ .

### A. Reduced basis

To make the method of the transfer matrix more powerful, it is useful to define a reduced basis  $||n\rangle_l$ , which classifies the allowed states according to the constraint on their preceding sites (to their left). The set of allowed states of length  $l$  can be classified according to a quantum number  $n$ ,  $1 \leq n \leq N$ . A state  $[\eta_1 \eta_2 \dots \eta_l]$ , which is an allowed state in the limit cycle with length  $l$ , is said to have the quantum number  $n$  if  $\eta_{N+1-n} = N$  and  $\eta_i < N$  for  $i < N + 1 - n$  when  $N + 1 - n \leq l$ , while  $\eta_i = l + n - 1$  and  $\eta_i < N$  for  $i < l$  when  $N + 1 - n > l$  [11]. A collection of such states can be denoted as  $||n\rangle_l$  and called a reduced state, while  $|n\rangle_l$  denotes a typical state in  $||n\rangle_l$ . Every allowed state belongs to one of the reduced states. We therefore call the set of  $||n\rangle_l$ , with  $1 \leq n \leq N$ , a reduced basis. The quantum number  $n$  labels the fact that, for any state  $[\eta_1 \eta_2 \dots \eta_l]$  in  $||n\rangle_l$ , one of the  $n$  slopes to the left of  $\eta_1$  has to be  $N$ .

For an allowed state of length  $L$   $[\sigma_1 \sigma_2 \dots \sigma_L]$ , the last  $l$  slopes  $[\sigma_{L-l+1} \sigma_{L-l+2} \dots \sigma_L]$  are an allowed state of length  $l$ , which belongs to some reduced basis  $||n\rangle_l$ , and  $[\sigma_1 \sigma_2 \dots \sigma_{L-l} \sigma'_{L-l+1} (= n)]$  is an allowed state of length  $L - l + 1$ . This correspondence is one to one for fixed  $l$  and this provides a route to construct the limit cycle of length  $L$  from the limit cycle of length  $l$  ( $< L$ ).

### B. The time average of space dependent functions

Let  $S_l$  be the number of allowed states of length  $l + 1$  with  $\sigma_{l+1} = N$ . From the three conditions for allowed states in the limit cycle, we have  $S_l = (N+1)^l$  for  $0 \leq l < N$ ,  $S_N = (N+1)^N - N^N$ , and  $S_{l+1} = (N+1)S_l - N^N S_{l-N}$  for  $l > N$ . For any allowed state of length  $L$  with  $\sigma_i = N$ ,  $[\sigma_{i+1} \sigma_{i+2} \dots \sigma_L]$  can be any allowed state of length  $L - i$ . The number of allowed states of length  $L$  with  $\sigma_i = N$  is thus  $N^{L-i} S_{i-1}$ .

It should be noted that the three conditions for allowed states in the limit cycle only differentiate  $\sigma_i = N$  and  $\sigma_i \neq N$  unless  $i = L$ . Since we know the total degeneracy of states for  $\sigma_i$  taking an arbitrary value and the degeneracy for states in which  $\sigma_i = N$  can be calculated using  $S_l$ , the degeneracy for  $\sigma_i \neq N$  can be easily obtained. Therefore  $S_l$  contains all the information about the time average of space dependent functions, such as the  $n$ -point function  $\langle \sigma_i \dots \sigma_j \rangle$ . Here  $\langle \dots \rangle$  means the average in time over a limit cycle. The one-point function can be evaluated in terms of the summation of all allowed states in a limit cycle, i.e.,

$$\begin{aligned} \langle \sigma_i \rangle &= \frac{NN^{L-i} S_{i-1} + \frac{N-1}{2} (N^L - N^{L-i} S_{i-1})}{N^L} \\ &= \frac{N-1}{2} + \frac{N+1}{2} \frac{S_{i-1}}{N^i}, \quad i < L. \end{aligned} \quad (2.1)$$

Similarly, the two-point function is

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle &= \frac{(N-1)^2}{4} + \frac{N^2-1}{4} \left( \frac{S_{i-1}}{N^i} + \frac{S_{j-1}}{N^j} \right) \\ &\quad + \frac{(N+1)^2}{4} \frac{S_{i-1} S_{j-i-1}}{N^j}, \quad i < j < L. \end{aligned} \quad (2.2)$$

Actually, all the  $n$ -point functions can be expressed in a similar form. Therefore the calculations of the  $n$ -point functions are reduced to the calculation of  $S_l$ . It should be noted that these time averages of space dependent functions are independent of  $L$ .

One can use mathematical induction to get the following closed form for  $S_l$ :

$$S_{mN+l} = (N+1)^{mN+l} + \sum_{j=1}^m (-1)^j (N+1)^{(m-j)N-j+l} N^{jN} \left[ \binom{(m-j)N+l}{j} + (N+1) \binom{(m-j)N+l}{j-1} \right] \quad (2.3)$$

with  $m \geq 0$  and  $0 \leq l < N$ . However, such a tedious closed form is difficult to use. A better form can be obtained by the transfer matrix method.

### C. Transfer matrix

For any state in  $|n\rangle_l$ , we can add a preceding slope  $\sigma = N$ , resulting in a state in  $|N\rangle_{l+1}$ , or add a preceding slope  $\sigma = 0, 1, 2, \dots, N-1$  if  $n > 1$ , resulting in a state in  $|n-1\rangle_{l+1}$ . We can express this property

in terms of a transfer operator  $T$  such that  $T||n\rangle_l = ||N\rangle_{l+1} + N||n-1\rangle_{l+1}$ , if  $n > 1$ , and  $T||1\rangle_l = ||N\rangle_{l+1}$ . For example, the first equation represents the fact that, given a typical state  $[\eta_1 \eta_2 \dots \eta_l]$  in  $||n\rangle_l$  the corresponding allowed state  $[\eta, \eta_1 \eta_2 \dots \eta_l]$  contains one state in  $||N\rangle_{l+1}$  and  $N$  states in  $||n-1\rangle_{l+1}$ . As far as the transfer matrix is concerned, all the allowed states within the same reduced state are identical and there is no need to distinguish them. Therefore they are counted as one in the calculation of the transfer matrix, which makes it easier to calculate  $S_l$ . The degeneracy associated with the

number of identical states in a reduced state can be easily included in the calculation of partition functions as shown in Eqs. (2.1) and (2.2). It is also clear from the definition of the transfer matrix that its properties are independent of the subscript  $l$ . Since it is obvious that each operation of  $T$  increases the  $l$  by 1, the subscript can be ignored completely.

Representing  $||N\rangle$  by  $\text{col}(1 \ 0 \ \dots \ 0)$ ,  $||N-1\rangle$  by  $\text{col}(0 \ 1 \ 0 \ \dots \ 0)$ , ...,  $||1\rangle$  by  $\text{col}(0 \ \dots \ 0 \ 1)$ , the transfer operator can thus be represented as an  $N \times N$  matrix:

$$T = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ N & 0 & 0 & \dots & 0 & 0 \\ 0 & N & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & N & 0 \end{pmatrix}. \quad (2.4)$$

Note that  $\text{col}(1 \ 1 \ \dots \ 1)$  is an eigenvector of  $T$  of eigenvalue  $N$ . The allowed states of length 1 are  $\sigma_1 = 1, 2, \dots, N$ , which correspond to the reduced basis  $||1\rangle_1, ||2\rangle_1, \dots, ||N\rangle_1$ , respectively. The number of allowed states of length  $L$  is easily proved to be  $(1 \ 1 \ \dots \ 1)T^{L-1}\text{col}(1 \ 1 \ \dots \ 1) = N^L$ . This is the simplest way of deriving this result, pointed out already in Ref. [8]. The number of states corresponding to the reduced basis  $|n\rangle_l$  with  $1 \leq n \leq N$  is  $(\delta_{N,n} \ \delta_{N-1,n} \ \dots \ \delta_{1,n})T^{l-1}\text{col}(1 \ 1 \ \dots \ 1) = N^{l-1}$ , which is independent of  $n$ , and

$$S_l = (1 \ 1 \ \dots \ 1)T^l\text{col}(1 \ 0 \ \dots \ 0). \quad (2.5)$$

$S_l$  can now be expressed in terms of the eigenvalues of the transfer matrix  $T$ , and we have

$$S_l = \frac{N^{l+1}}{N+1} \left( 2 + \sum_{m=1}^{N-1} \lambda_m^{i+1} \right), \quad (2.6)$$

where  $N\lambda_0 (\lambda_0 = 1)$ ,  $N\lambda_1, \dots, N\lambda_{N-1}$  are the eigenvalues of  $T$  and  $\lambda_1, \lambda_2, \dots, \lambda_{N-1}$  are the solutions of the equation

$$1 + 2\lambda + 3\lambda^2 + \dots + N\lambda^{N-1} = 0. \quad (2.7)$$

The one- and two-point functions are thus

$$\langle \sigma_i \rangle = \frac{N+1}{2} - \frac{1}{2} \sum_{m=1}^{N-1} \lambda_m^i, \quad i < L \quad (2.8)$$

and

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle &= \frac{(N+1)^2}{4} + \frac{N+1}{4} \sum_{m=1}^{N-1} \lambda_m^i + \frac{N-1}{4} \sum_{m=1}^{N-1} \lambda_m^j \\ &\quad + \frac{1}{2} \sum_{m=1}^{N-1} \lambda_m^{j-i} + \frac{1}{4} \sum_{m=1}^{N-1} \lambda_m^i \sum_{k=1}^{N-1} \lambda_k^{j-i}, \\ &\quad i < j < L. \end{aligned} \quad (2.9)$$

For  $N = 2$  and  $3$ , the results agree with [8,10].

With  $y = 1/\lambda$  and multiplying Eq. (2.7) by  $y - 1$ , we have

$$y^N + y^{N-1} + \dots + y = N \Rightarrow |y| > 1 \Rightarrow |\lambda| < 1, \quad (2.10)$$

where the redundant solution  $y = 1$  is excluded. Multiplying by  $y - 1$  again, we have

$$y^{N+1} = (N+1)y - N \Rightarrow (N+1)|y| + N \geq |y|^{N+1}, \quad (2.11)$$

which implies  $|y| \leq 2$  for  $N \geq 2$  and therefore  $|y| \leq (3N+2)^{1/(N+1)}$ . The solutions of  $y$  are inside the annulus  $1 < |y| \leq y_{max} = (3N+2)^{1/(N+1)}$ . In fact for large  $N$  one can show that  $y_{max} \cong 1 + N^{-1} \ln(2N)$  and therefore as  $N \rightarrow \infty$  all the roots have  $|y|$  near 1. From Eq. (2.11) we have

$$|y|^{N+1} = (N+1) \left| y - \frac{N}{N+1} \right|,$$

which means the solutions with smaller  $|y|$  are all near  $y = 1$  for large  $N$ .

To explore the asymptotic behavior of  $S_l$ , we are interested in the solutions of Eq. (2.7) with larger  $|\lambda|$ . Therefore we would like to find the solutions of Eq. (2.11) with smaller  $|y|$  excluding the redundant solution  $y = 1$ . Considering only one of the complex conjugate solutions in the upper half of the complex plane, let  $y_j = a_j e^{i\theta_j}$  with  $0 < j < N/2$  and  $\phi_j = \angle[(N+1)y_j - N]$  with  $0 < \phi_j < \pi$  such that  $(N+1)\theta_j = \phi_j + 2j\pi$ . It should be noted that  $(N+1)\theta_0 = \phi_0$  results in the redundant solution  $y_0 = 1$ . From Eq. (2.11) we have

$$a_j^{2N+2} = (N+1)^2 a_j^2 + N^2 - 2N(N+1)a_j \cos \theta_j$$

and

$$\tan \phi_j = \frac{(N+1)a_j \sin \theta_j}{(N+1)a_j \cos \theta_j - N}.$$

For large  $N$  we only have to consider the case of  $j/N \ll 1$  to find the smaller  $|y|$  solutions. Letting  $a_j = 1 + x_j/(N+1)$  and  $\theta_j = (2j\pi + \phi_j)/(N+1) \ll 1$ , we have (as  $N \rightarrow \infty$ )

$$e^{2x_j} - (1 + x_j)^2 \cong (\phi_j + 2j\pi)^2 \quad (2.12)$$

and

$$\tan \phi_j \cong \frac{2j\pi + \phi_j}{1 + x_j}, \quad (2.13)$$

where the trigonometric functions have been expanded to the second order. Table I shows the values of the first few  $x$ 's and  $\phi$ 's as  $N \rightarrow \infty$ . From Eq. (2.9), the correlation length is found to be  $N/x_1$  for large  $N$ .

TABLE I. Some numerical values of  $x_j$  and  $\phi_j$  as  $N \rightarrow \infty$ .

$j$	$x_j$	$\phi_j$
1	2.089	1.178
2	2.664	1.312
3	3.026	1.374
4	3.292	1.410
5	3.501	1.435

### III. MULTIFRACTAL CHARACTERIZATION OF THE LIMIT CYCLE

Now we would like to describe the limit cycle (attractor) quantitatively. In general, the state in the limit cycle corresponds to a path in a tree. It is now possible to identify a multifractal along the time axis. Rescaling the time variable so that each time step is of length  $\delta = N^{-L}$ , the duration for a limit cycle fits into the time interval  $[0, 1]$ . Consider the states  $[\sigma_1 \sigma_2 \dots \sigma_L]$  in the limit cycle which contains  $\mathbf{N}_0$  0's,  $\mathbf{N}_1$  1's, ..., and  $\mathbf{N}_N$   $N$ 's. The corresponding time steps for such states form a fractal in the interval  $[0, 1]$  with fractal dimension  $f(\xi_0, \xi_1, \dots, \xi_N)$  where  $\xi_i = \mathbf{N}_i/L$ , for  $0 \leq i \leq N$  and satisfy  $\sum_i \xi_i = 1$ . The above relations are exact as  $L \rightarrow \infty$ .

For  $N = 2$ , it is straightforward to get

$$f(\xi_0, \xi_1, \xi_2) = \frac{1}{\ln 2} \left[ \xi_0 \ln \left( \frac{\xi_2 - \xi_1 - \xi_0}{\xi_0} \right) + \xi_1 \ln \left( \frac{\xi_2 - \xi_1 - \xi_0}{\xi_1} \right) + \xi_2 \ln \left( \frac{\xi_2}{\xi_2 - \xi_1 - \xi_0} \right) \right]. \quad (3.1)$$

This function depends on  $\xi_0$  and  $\xi_1$  in the same way, which suggests that the boundary effect due to the specific constraint on  $\sigma_L$  is negligible in the large  $L$  limit. Neglecting this boundary effect, there is no more difference between  $n = 0, 1, \dots$ , and  $N-1$ . Thus it is natural to define the fractal dimension as a function of  $\xi = \mathbf{N}_N/L$  with  $1/N \leq \xi \leq 1$ . The problem is equivalent to counting the ways to distribute  $(1-\xi)L$  objects into  $\xi L$  slots and every slot admits at most  $N-1$  objects. Now we have

$$(N^L)^{f(\xi)} \cong \frac{1}{(L - \mathbf{N}_N)!} \frac{d^{L - \mathbf{N}_N}}{dx^{L - \mathbf{N}_N}} \left( \frac{x^N - 1}{x - 1} \right)^{\mathbf{N}_N} \Big|_{x=0} \times N^{L - \mathbf{N}_N} \quad (3.2)$$

The “ $\cong$ ” is “ $=$ ” as  $L \rightarrow \infty$ . After taking the differentiation with  $x$  separately on  $[(x^N - 1)/(x - 1)]^{\mathbf{N}_N}$  and  $1 + x + x^2 + \dots + x^{N-1}$ , we have

$$N^{f + (1-\xi)f'} = \sum_{k=0}^{N-1} N^{k(1-f) + k\xi f'} \quad (3.3)$$

where  $f(\xi + \Delta\xi)$  is approximated as  $f(\xi) + \Delta\xi f'$ . Let  $f(\xi) = g(\xi)/\ln N + 1 - \xi$  with  $R \equiv e^{g'}$  and  $Z \equiv e^{-(g - \xi g')}$ , which means

$$\frac{R'}{R} \xi = \frac{Z'}{Z}. \quad (3.4)$$

Now Eq. (3.3) becomes

$$R = Z + Z^2 + \dots + Z^N. \quad (3.5)$$

It follows from the definitions of  $R$  and  $Z$  that

$$g(\xi) = \xi \ln(Z + Z^2 + \dots + Z^N) - \ln Z \quad (3.6)$$

with

$$\xi = \frac{1 + Z + Z^2 + \dots + Z^{N-1}}{1 + 2Z + 3Z^2 + \dots + NZ^{N-1}}. \quad (3.7)$$

This result obviously satisfies the boundary conditions  $f(\xi = 1/N) = 1 - 1/N$  and  $f(\xi = 1) = 0$ .  $f(\xi) = 1$  and  $f'(\xi) = 0$  happen at  $\xi = 2/(N+1)$ , which is consistent with  $\langle \sigma_i \rangle \rightarrow (N+1)/2$  as  $i \rightarrow \infty$  from Eq. (2.8). Equations (3.6) and (3.7) express the fractal dimension  $f(\xi)$  in terms of the variable  $Z$ ,  $0 \leq Z \leq \infty$ .

### IV. DISCUSSIONS

We have introduced the concept of the transfer matrix into the study of steady properties of sandpile models. This is closely related to the Hamiltonian formulation of the usual statistical mechanics. Indeed, one may regard the formulas for the correlation function such as Eqs. (2.1) and (2.2) as “path integral” expressions in a discrete formulation. The transfer matrix plays the role of the evolution operator.

The usefulness of the transfer matrix formulation was illustrated by deriving the one- and two-point correlation functions for a deterministic sandpile model with an arbitrary critical slope  $N$ . We found that the two-point function decreases exponentially as the separation of the two points increases with a correlation length depending on  $N$ . In units of the lattice spacing, the correlation length is found to be the largest (in magnitude) root of an  $(N-1)$ th order polynomial. The correlation length is proportional to  $N$  in the large  $N$  limit.

In the  $L \rightarrow \infty$  limit, the states in the limit cycle can be characterized as a multifractal. We derive an explicit expression for the multifractal dimension for arbitrary  $N$ . It was pointed out in Ref. [8] that the multifractal structure reflects the spatial-temporal organization of the model and is expected to be found in any deterministic model of self-organized criticality. This multifractal structure indicates that, even though there exists only short range spatial correlation for any  $N$ , power law behavior is expected for the time-time correlation function. It would be interesting to investigate the relation between this correlation function and the multifractal nature of the attractor.

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